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General solution to the spherical Raman–Nath equation

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Abstract. The spherical Raman–Nath equation, a difference-differential equation describing stimulated Compton scattering, is transformed into a partial differential equation for the probability amplitude in spherical phase space by using the Q representation of the atomic coherent states. With the quantum electron recoil as the perturbation parameter, a perturbative solution up to indefinite high order is obtained. This is a general solution because the initial condition can be arbitrary.

1. Introduction

Stimulated Compton scattering (SCS) is the fundamental process in a free-electron laser (FEL) working in the Compton regime. The Hamiltonian for SCS in a moving frame, in which the frequency of the radiation propagating along one direction is identical to that propagating in the opposite direction, can be written as

$$H = p^2/2m + \hbar\omega(a_f^\dagger a_f + a_b^\dagger a_b) + \hbar\Lambda[a_f^\dagger a_b \exp(-2ikz) + a_b^\dagger a_f \exp(2ikz)] \tag{1.1}$$

where p is the operator representing the momentum of the electron, $a_f^\dagger$ ($a_b^\dagger$) is the creation operator of the forward (backward) propagating field, Λ is the coupling constant and $\hbar k = \hbar\omega/c$ is the momentum of a photon. This Hamiltonian implies the conservation of total photon number N and total linear momentum.

We assume that the initial quantum state of the complete system can be written as the product of a plane wave of momentum p_0 for the electron, a Fock state of n_0 photons for the forward propagating radiation, and a Fock state of $N - n_0$ photons for the backward propagating radiation, namely

$$|\psi_0\rangle \equiv K_0 \exp(ip_0 z/\hbar)|n_0\rangle_f |N - n_0\rangle_b \tag{1.2}$$

where K_0 is a normalisation constant. Because of the conservation of total linear momentum and total photon number, the quantum states evolving from the initial state $|\psi_0\rangle$ can always be written as linear combinations of the $N + 1$ basic states, each being sufficiently identified by a single quantum number n, the photon number in the forward propagating field, denoted by

$$|n\rangle \equiv K_n \exp\{i[p_0/\hbar - 2(n - n_0)k]z\}|n\rangle_f |N - n\rangle_b \tag{1.3}$$

where K_n again is a normalisation constant. In terms of these basic states, the time-dependent quantum state of the system can be written as

$$|\psi(t)\rangle = \exp\{-i[(p_0 + 2n_0\hbar k)^2/2m\hbar + N\omega]t\} \sum_{n=0}^{N} C_n(t)|n\rangle. \tag{1.4}$$

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Substituting (1.4) into the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle \tag{1.5}$$

with the Hamiltonian given by (1.1), we obtain the following difference-differential equation (Dattoli and Renieri 1985):

$$i\frac{d}{dt} C_n(t) = (-2n\Delta + n^2 E)C_n(t) + \Lambda[(N-n)(n+1)]^{1/2} C_{n+1}(t) + \Lambda[(N-n+1)n]^{1/2} C_{n-1}(t) \tag{1.6}$$

where $C_n(t)$ is the probability amplitude that n photons are propagating forward and $N-n$ photons are propagating backward at time t, and

$$\Delta \equiv k(p_0 + 2n_0 \hbar k)/m \qquad E \equiv 2\hbar k^2/m \tag{1.7}$$

are two constants, the latter being related to the quantum recoil of the electron.

Equation (1.6) has been recognised by Bosco *et al* (1984) as one of the various types of generalised Raman–Nath equations (RNE). Hence these authors call it spherical RNE. The original RNE was derived by Raman and Nath (1937) to describe light diffraction by ultrasound. The various types of RNE appear in a large number of physical problems, as pointed out by Bosco and Dattoli (1983). They are all difficult to solve because of the existence of the non-linear term n^2E.

Bosco *et al* (1984) have obtained solutions to (1.6) under the simplifying assumption that $E=0$. We have previously obtained (Lee 1985a) a perturbative solution to first order in the perturbation parameter E. More recently (Lee 1985b), we used the Q representation of atomic coherent states (Arecchi *et al* 1972) to transform the spherical RNE into a partial differential equation for the probability density function over the spherical phase space, and obtained a perturbative solution up to arbitrarily high order in E, with all the photons propagating initially in one direction only. The trouble with this last solution is that the analytic expression is so lengthy that it is very difficult to use it to calculate any observable physical quantities.

We have now realised that it is much easier to deal with the probability amplitude than with the probability density function. In the following, we will derive the partial differential equation for the probability amplitude in a spherical phase space and find a general perturbative solution up to indefinite high order in the perturbation parameter E.

2. Partial differential equation in spherical phase space

Coherent states have proved to be very powerful mathematical tools for analysing quantum optical systems. The present paper will serve as yet another in the long list of such examples. The two most popular kinds of coherent states are the Glauber coherent states (Glauber 1963a, b) for harmonic oscillators and the atomic coherent states for angular momentum or two-level atom systems. If the range of the photon number in a radiation field is from 0 to ∞, then the natural choice of coherent state representation will be that of Glauber coherent states, which have a Poisson distribution. In the system of this paper, the range is from 0 to N, so the $N+1$ possible photon states should be identified with the $N+1$ possible angular momentum states with fixed total angular momentum $J \equiv N/2$, and the choice of coherent-state representation should be that of atomic coherent states, which have a binomial distribution. The phase space for Glauber coherent states is the complex plane and that for atomic coherent states is the spherical surface.

The density matrix to be constructed from the solution of (1.6) is of the following form:

$$\rho(t) = \sum_{n=0}^{N} \sum_{m=0}^{N} C_m^*(t) C_n(t) |n\rangle\langle m|. \tag{2.1}$$

The atomic coherent states are defined in terms of $|n\rangle$ as

$$|\theta, \phi\rangle \equiv \sum_{n=0}^{N} |n\rangle \binom{N}{n}^{1/2} (\cos \tfrac{1}{2}\theta)^{N-n} (\sin \tfrac{1}{2}\theta\, e^{-i\phi})^n \tag{2.2}$$

and the probability density function over the spherical surface in the Q representation is defined as

$$Q(\theta, \phi, t) \equiv \langle \theta, \phi | \rho(t) | \theta, \phi \rangle \equiv P^*(\theta, \phi, t) P(\theta, \phi, t) \tag{2.3}$$

where

$$P(\theta, \phi, t) = \sum_{n=0}^{N} \binom{N}{n}^{1/2} C_n(t) (\cos \tfrac{1}{2}\theta)^{N-n} (\sin \tfrac{1}{2}\theta\, e^{i\phi})^n \tag{2.4}$$

is the probability amplitude. The normalisation condition is

$$\frac{N+1}{4\pi} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi\, Q(\theta, \phi, t) = 1. \tag{2.5}$$

It will be convenient to introduce the following dimensionless parameters:

$$\delta \equiv \Delta/\Omega \qquad \lambda \equiv \Lambda/\Omega \qquad \varepsilon \equiv NE/\Omega \qquad \tau \equiv 2\Omega t \tag{2.6}$$

where

$$\Omega \equiv (\Delta^2 + \Lambda^2)^{1/2} \tag{2.7}$$

which implies that $\delta^2 + \lambda^2 = 1$.

Using (1.6) and (2.4), we can obtain the partial differential equation for $P(\theta, \phi, \tau)$ as

$$\left(i \frac{\partial}{\partial \tau} + i(\lambda \cot \tfrac{1}{2}\theta \cos\phi - \delta) \frac{\partial}{\partial \phi} + \lambda\, e^{i\phi} \frac{\partial}{\partial \theta} + \frac{\varepsilon}{2N} \frac{\partial^2}{\partial \phi^2} \right) P(\theta, \phi, \tau) = 0. \tag{2.8}$$

The connection between (1.5) and (2.8) can also be easily established by using the following table of corresponding operators in the two different spaces:

$\|\psi(t)\rangle$	$P(\theta, \phi, \tau)$
p	$p_0 + 2\hbar k \left(n_0 + i \frac{\partial}{\partial \phi} \right)$
$a_f^\dagger a_b \exp(-2ikz)$	$-e^{i\phi} \left(2 \frac{\partial}{\partial \theta} + i \cot \frac{\theta}{2} \frac{\partial}{\partial \phi} \right)$
$a_b^\dagger a_f \exp(2ikz)$	$-i\, e^{-i\phi} \cot \frac{\theta}{2} \frac{\partial}{\partial \phi}$

3. Perturbative solution with definite initial photon numbers

In this section, we try to find a perturbative solution to (2.8) with ε as the perturbation parameter. We only consider the simple initial condition that there are n_0 photons propagating in one direction and $N-n_0$ photons propagating in the opposite direction, i.e.

$$C_n(0)=\delta_{n,n_0}. \tag{3.1}$$

Substituting (3.1) into (2.4), we obtain

$$P_{n_0}(\theta,\phi,0)=\binom{N}{n_0}^{1/2}(\cos\tfrac{1}{2}\theta)^{N-n_0}(\sin\tfrac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi})^{n_0} \tag{3.2}$$

as the initial condition for (2.8). It is convenient to consider two different situations separately.

3.1. $n_0 \ll N$

We try a perturbative solution to (2.8) of the following form:

$$P_{n_0}(\theta,\phi,\tau)\equiv\exp(\mathrm{i}N\delta\tau/2)\binom{N}{n_0}^{1/2}$$
$$\times(R_0(\theta,\phi,\tau)+\varepsilon R_1(\theta,\phi,\tau)+\varepsilon^2R_2(\theta,\phi,\tau)+\ldots) \tag{3.3}$$

with the initial conditions

$$R_0(\theta,\phi,0)=(\cos\tfrac{1}{2}\theta)^{N-n_0}(\sin\tfrac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi})^{n_0} \tag{3.4a}$$

$$R_{l>0}(\theta,\phi,0)=0. \tag{3.4b}$$

Substitution of (3.3) into (2.8) yields

$$\hat{D}_0R_0(\theta,\phi,\tau)=0 \tag{3.5a}$$

$$\hat{D}_0R_{l+1}(\theta,\phi,\tau)+\hat{D}_1R_l(\theta,\phi,\tau)=0 \tag{3.5b}$$

where

$$\hat{D}_0\equiv\mathrm{i}\frac{\partial}{\partial\tau}+\mathrm{i}(\lambda\cot\tfrac{1}{2}\theta\cos\phi-\delta)\frac{\partial}{\partial\phi}+\lambda\,\mathrm{e}^{\mathrm{i}\phi}\frac{\partial}{\partial\theta}-\delta \tag{3.6a}$$

$$\hat{D}_1\equiv\frac{1}{2N}\frac{\partial^2}{\partial\phi^2}. \tag{3.6b}$$

The solution to (3.5a) satisfying the initial condition (3.4a) can be written as

$$R_0(\theta,\phi,\tau)=[F_1(\theta,\phi,\tau)]^{N-n_0}[F_2(\theta,\phi,\tau)]^{n_0} \tag{3.7}$$

where

$$F_1(\theta,\phi,\tau)\equiv(\cos\tfrac{1}{2}\tau-\mathrm{i}\delta\sin\tfrac{1}{2}\tau)\cos\tfrac{1}{2}\theta-\mathrm{i}\lambda\sin\tfrac{1}{2}\tau\sin\tfrac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi} \tag{3.8a}$$

$$F_2(\theta,\phi,\tau)\equiv-\mathrm{i}\lambda\sin\tfrac{1}{2}\tau\cos\tfrac{1}{2}\theta+(\cos\tfrac{1}{2}\tau+\mathrm{i}\delta\sin\tfrac{1}{2}\tau)\sin\tfrac{1}{2}\theta\,\mathrm{e}^{\mathrm{i}\phi}. \tag{3.8b}$$

As long as n_0 and l are both much less than N, the solution to (3.5b) can be approximated by the expression

$$R_l(\theta, \phi, \tau) = \binom{(N-n_0)/2}{l} [F_1(\theta, \phi, \tau)]^{N-n_0-2l} [G_1(\theta, \phi, \tau)]^l [F_2(\theta, \phi, \tau)]^{n_0} \tag{3.9}$$

where $G_1(\theta, \phi, \tau)$ must satisfy the equation

$$\hat{D}_0 G_1(\theta, \phi, \tau) = \left(\frac{\partial}{\partial \phi} F_1(\theta, \phi, \tau)\right)^2 = (\lambda \sin \tfrac{1}{2}\tau \sin \tfrac{1}{2}\theta\, e^{i\phi})^2. \tag{3.10}$$

Let us try a solution to (3.10) of the following form:

$$G_1(\theta, \phi, \tau) = g_1(\tau) \cos^2 \tfrac{1}{2}\theta + g_2(\tau) \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta\, e^{i\phi} + g_3(\tau) \sin^2 \tfrac{1}{2}\theta\, e^{2i\phi}. \tag{3.11}$$

Substitution of (3.11) into (3.10) yields the simultaneous equations

$$\begin{aligned} &dg_1/d\tau + i\delta g_1 + i\tfrac{1}{2}\lambda g_2 = 0 \\ &dg_2/d\tau + i\lambda g_1 + i\lambda g_3 = 0 \\ &dg_3/d\tau - i\delta g_3 + i\tfrac{1}{2}\lambda g_2 = -i\lambda^2 \sin^2 \tau/2. \end{aligned} \tag{3.12}$$

The solution to (3.12) satisfying the initial condition (3.4*b*) can be expressed as follows:

$$g_1(\tau) = -i(\lambda^4/8) h(\tau) \tag{3.13a}$$

$$g_2(\tau) = (\lambda^3/4) h'(\tau) + i(\lambda^3 \delta/4) h(\tau) \tag{3.13b}$$

$$g_3(\tau) = -(\lambda^2 \delta/4) h'(\tau) + i[(\lambda^4/8) h(\tau) + (\lambda^2/4) h''(\tau)] \tag{3.13c}$$

where

$$h(\tau) = 3 \sin \tau - \tau \cos \tau - 2\tau. \tag{3.14}$$

$h'(\tau)$ is the first derivative of $h(\tau)$, and $h''(\tau)$ is the second derivative.

We can now put the components of the solution together. Substitution of (3.9) into (3.3) gives

$$P_{n_0}(\theta, \phi, \tau) = \exp(iN\delta\tau/2) \binom{N}{n_0}^{1/2} \times \{[F_1(\theta, \phi, \tau)]^2 + \varepsilon G_1(\theta, \phi, \tau)\}^{(N-n_0)/2} [F_2(\theta, \phi, \tau)]^{n_0} \tag{3.15}$$

where $F_1(\theta, \phi, \tau)$ and $F_2(\theta, \phi, \tau)$ are given in (3.8) and the explicit expression for $G_1(\theta, \phi, \tau)$ can be obtained by using (3.14) in (3.13) and then substituting (3.13) into (3.11) with the result

$$\begin{aligned} G_1(\theta, \phi, \tau) = {} & [-i\tfrac{1}{8}\lambda^4 (3 \sin \tau - \tau \cos \tau - 2\tau)] \cos^2 \tfrac{1}{2}\theta \\ & + [\tfrac{1}{4}\lambda^3 (2 \cos \tau + \tau \sin \tau - 2) + i\tfrac{1}{4}\lambda^3 \delta (3 \sin \tau - \tau \cos \tau - 2\tau)] \\ & \times \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta\, e^{i\phi} + \{-\tfrac{1}{4}\lambda^2 \delta (2 \cos \tau + \tau \sin \tau - 2) \\ & + i\tfrac{1}{8}\lambda^2 [(3\lambda^2 - 2) \sin \tau - (\lambda^2 - 2)\tau \cos \tau - 2\lambda^2 \tau]\} \sin^2 \tfrac{1}{2}\theta\, e^{2i\phi}. \end{aligned} \tag{3.16}$$

3.2. $n_0 \sim N$

We now consider the case when n_0 is of the same order as N which is assumed to be much greater than 1. The solution to (2.8) under this condition can be approximated by the expression

$$P_{n_0}(\theta, \phi, \tau) = \exp(\mathrm{i}N\delta\tau/2)\binom{N}{n_0}^{1/2}\{[F_1(\theta, \phi, \tau)]^2 + \varepsilon G_1(\theta, \phi, \tau)\}^{(N-n_0)/2}$$

$$\times\{[F_2(\theta, \phi, \tau)]^2 + \varepsilon G_2(\theta, \phi, \tau)\}^{n_0/2} \qquad (3.17)$$

where $F_1(\theta, \phi, \tau)$, $G_1(\theta, \phi, \tau)$ and $F_2(\theta, \phi, \tau)$ have been given in the previous subsection, and $G_2(\theta, \phi, \tau)$ must satisfy the equation

$$\hat{D}_0 G_2(\theta, \phi, \tau) = \left(\frac{\partial}{\partial\phi} F_2(\theta, \phi, \tau)\right)^2$$

$$= -[\tfrac{1}{2}\lambda^2 + (1 - \tfrac{1}{2}\lambda^2)\cos\tau + \mathrm{i}\delta\sin\tau]\sin^2\tfrac{1}{2}\theta\, \mathrm{e}^{2\mathrm{i}\phi}. \qquad (3.18)$$

The solution to (3.18) can be obtained by a similar procedure to that used in solving (3.10). Without repeating the parallel description, we simply present the final explicit expression for $G_2(\theta, \phi, \tau)$ as follows:

$$G_2(\theta, \phi, \tau) = \{-\tfrac{1}{4}\lambda^2\delta(2\cos\tau + \tau\sin\tau - 2)$$

$$+\mathrm{i}\tfrac{1}{8}\lambda^2[(3\lambda^2 - 2)\sin\tau - (\lambda^2 - 2)\tau\cos\tau - 2\lambda^2\tau]\}\cos^2\tfrac{1}{2}\theta$$

$$+\{\tfrac{1}{4}\lambda[(4 - 6\lambda^2)(\cos\tau - 1) + (4 - 3\lambda^2)\tau\sin\tau]$$

$$+\mathrm{i}\tfrac{1}{4}\lambda\delta[(4 - 3\lambda^2)\sin\tau - (4 - \lambda^2)\tau\cos\tau + 2\lambda^2\tau]\}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}\phi}$$

$$+\{\delta[\lambda^2(\cos\tau - 1) - (1 - \lambda^2/2)\tau\sin\tau]$$

$$+\mathrm{i}[\lambda^2(1 - 3\lambda^2/8)\sin\tau + (1 - \lambda^2 + \lambda^4/8)\tau\cos\tau + \lambda^4\tau/4]\}\sin^2\tfrac{1}{2}\theta\, \mathrm{e}^{2\mathrm{i}\phi}.$$

$$(3.19)$$

4. Summary

By using the Q representation of atomic coherent states, we have transformed the spherical Raman-Nath equation, which describes stimulated Compton scattering, into a partial differential equation for the probability amplitude in spherical phase space. Using the quantum recoil of the electron as the perturbation parameter, we have obtained a closed form solution up to indefinite high order, assuming that the total number of photons involved in the scattering is much greater than one.

In a previous paper, we obtained the solution to the spherical Raman-Nath equation under a very specific condition, namely that all the photons are initially propagating in one direction only. The main achievement of the present paper is that the initial photon numbers propagating in both directions can be arbitrary. Another improvement is that the solutions are in the form of a probability amplitude rather than a probability density function, the former having much simpler analytical expressions.

A natural application of the solution obtained in this paper is to calculate the evolution of the photon statistics resulting from stimulated Compton scattering. This will help us understand the coherent properties of free-electron lasers.

We speculate that other types of Raman-Nath equations can be solved by a similar approach to that adopted in this paper.

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References

Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211
Bosco P and Dattoli G 1983 *J. Phys. A: Math. Gen.* **16** 4409
Bosco P, Gallardo J and Dattoli G 1984 *J. Phys. A: Math. Gen.* **17** 2739
Dattoli G and Renieri A 1985 *Laser Handbook* vol 4, ed M L Stitch and M S Bass (Amsterdam: North-Holland)
Glauber R J 1963a *Phys. Rev.* **130** 2529
—— 1963b *Phys. Rev.* **131** 2766
Lee C T 1985a *Phys. Rev.* A **31** 1212
—— 1985b *J. Phys. A: Math. Gen.* **18** L1139
Raman C W and Nath N S 1937 *Proc. Ind. Acad. Sci.* **2** 406